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UNIFORMLY SHADOWING PROPERTY

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Let M be a closed C^∞ manifold and $C^r(M)$ be the set of all C^r-differentiable maps endowed with the C^r-topology ($r \geq 1$). D_xf is the derivative of f at x. Denote as $\tilde{M}$ the topological product space $\prod_{-\infty}^{\infty} M$ and define a compatible metric $\tilde{d}$ on $\tilde{M}$ by $\tilde{d}((x_n),(y_n)) = \sum_{-\infty}^{\infty} d(x_n,y_n)/2^{|n|}$ for $(x_n),(y_n) \in \tilde{M}$, where d is a metric on M induced by a Riemannian metric. We define a continuous map $\tilde{f} : \tilde{M} \to \tilde{M}$ by

$$\tilde{f}((x_n)) = (f(x_n)).$$

Then the projection $P^0 : \tilde{M} \to M$ defined by $P^0((x_n)) = x_0$ satisfies $P^0 \circ \tilde{f} = f \circ P^0$. For a subset Λ an $\tilde{f}$-invariant set Λ_f is defined by

$$\Lambda_f = \{(x_n) \in \tilde{M} : x_n \in \Lambda,\ f(x_n) = x_{n+1},\ n \in \mathbb{Z}\}.$$

If $\Lambda_f \neq \emptyset$ then $\tilde{f}|\Lambda_f : \Lambda_f \to \Lambda_f$ is a surjective homeomorphism. Remark that $\Lambda_f = M_f \neq \emptyset$ when $\Lambda = M$. We say that each element of M_f is an orbit of f.

For $\delta \geq 0$ a sequence $\{x^i\}_{i\in\mathbb{Z}} \subset M$ is called a *δ-pseudo-orbit* of f if $d(f(x^i), x^{i+1}) \leq \delta$ for every $i \in \mathbb{Z}$. A sequence $\{x^i\}_{i\in\mathbb{Z}} \subset M$ is said to be *ε-traced* by an orbit $(y_i) \in M_f$ if $d(x^i, y_i) < \varepsilon$ for every $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-traced by an orbit of f.

A sequence $\{\tilde{x}^i\}_{i\in\mathbb{Z}} \subset M_f$ is called an orbit of $\tilde{f}$ if $\tilde{f}(\tilde{x}^i) = \tilde{x}^{i+1}$. For $\delta \geq 0$ a sequence $\{\tilde{x}^i\}_{i\in\mathbb{Z}} \subset M_f$ is a *δ-pseudo-orbit* of $\tilde{f}$ if $\tilde{d}(\tilde{f}(\tilde{x}^i), \tilde{x}^{i+1}) \leq \delta$ for every $i \in \mathbb{Z}$. A sequence $\{\tilde{x}^i\}_{i\in\mathbb{Z}} \subset \tilde{M}$ is said to be *ε-traced* by an orbit $(\tilde{y}^i)$ of $\tilde{f}$ if $\tilde{d}(\tilde{x}^i, \tilde{y}^i) < \varepsilon$ for every $i \in \mathbb{Z}$. We say that $\tilde{f}$ satisfies the *shadowing property* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit of $\tilde{f}$ can be ε-traced by an orbit of $\tilde{f}$.

We say that $\tilde{f}$ satisfies C^r *uniformly shadowing property* if there is a neighborhood $\mathcal{U}(f)$ of f in $C^r(M)$ with the property that for $\varepsilon > 0$ there is $\delta > 0$ such that for $g \in \mathcal{U}(f)$ every δ-pseudo-orbit of $\tilde{g}$ is ε-traced by an orbit of $\tilde{g}$.

Let $\pi : TM \to M$ be a tangent bundle of M and $\|\cdot\|$ be a Riemannian metric on TM. Define a subset of the product topological space $\tilde{M} \times TM$ by

$$T\tilde{M} = \{(\tilde{x}, v) \in \tilde{M} \times TM : P^0(\tilde{x}) = \pi(v)\}$$

and define a Finsler $||\cdot||$ on $T\tilde{M}$ by $||(\tilde{x},v)||=||v||$. Then $\tilde{\pi}:T\tilde{M}\to\tilde{M}$ defined by $\tilde{\pi}(\tilde{x},v)=\tilde{x}$ is a C^0-vector bundle over $\tilde{M}$. Define the projection $\bar{P}^0:T\tilde{M}\to TM$ by $\bar{P}^0(\tilde{x},v)=v$. Then,

$$\bar{P}^0|T_{\tilde{x}}\tilde{M}:T_{\tilde{x}}\tilde{M}\to T_{P^0(\tilde{x})}M$$

is a linear isomorphism where $T_{\tilde{x}}\tilde{M}=\tilde{\pi}^{-1}(\tilde{x})$. A linear bundle map $D\tilde{f}:T\tilde{M}\to T\tilde{M}$ covering $\tilde{f}$ is defined by

$$D\tilde{f}(\tilde{x},v)=(\tilde{f}(\tilde{x}),D_{P^0(\tilde{x})}f(v)).$$

Then we have $D\tilde{f}(T_{\tilde{x}}\tilde{M})\subset T_{\tilde{f}(\tilde{x})}\tilde{M}$ and $\bar{P}^0\circ D\tilde{f}=Df\circ\bar{P}^0$. To simplify the notation we write $D_{\tilde{x}}\tilde{f}=D\tilde{f}|T_{\tilde{x}}\tilde{M}$. For a subset $\tilde{\Lambda}$ define

$$T\tilde{M}|\tilde{\Lambda}=\bigcup_{\tilde{x}\in\tilde{\Lambda}}T_{\tilde{x}}\tilde{M}.$$

A closed f-invariant set Λ $(f(\Lambda)=\Lambda)$ is said to be *hyperbolic* if $T\tilde{M}|\Lambda_f$ splits into the Whitney sum $T\tilde{M}|\Lambda_f=\mathbb{E}^s\oplus\mathbb{E}^u$ of subbundles $\mathbb{E}^s$ and $\mathbb{E}^u$, and there are $C>0$ and $0<\lambda<1$ such that

(i) $D\tilde{f}(\mathbb{E}^s)\subset\mathbb{E}^s$ and $D\tilde{f}(\mathbb{E}^u)=\mathbb{E}^u$,

(ii) $D\tilde{f}|\mathbb{E}^u:\mathbb{E}^u\to\mathbb{E}^u$ is invertible,

(iii) $||D\tilde{f}^n|\mathbb{E}^s||\le C\lambda^n$ and $||(D\tilde{f}|\mathbb{E}^u)^{-n}||\le C\lambda^n$ for $n\ge 0$,

where $||\ T\ ||$ denotes the supremum norm of a linear bundle map T. The number λ is called the *skewness* of the hyperbolic set Λ. For $\varepsilon>0$ and

$\tilde{x} \in M_f$ the *local stable* and the *local unstable manifolds* are defined by

$$\begin{aligned} W^s_\varepsilon(\tilde{x}, f) &= \{y \in M : d(x_n, f^n(y)) \le \varepsilon \text{ for } n \ge 0\}, \\ W^u_\varepsilon(\tilde{x}, f) &= \left\{ y \in M \;\middle|\; \begin{array}{l} \text{there exists } \tilde{y} \in M_f \text{ such that } y_0 = y \\ \text{and } \ d(x_{-n}, y_{-n}) \le \varepsilon \text{ for } n \ge 0 \end{array} \right\}. \end{aligned}$$

Then, $W^s_\varepsilon(\tilde{x}, f) = W^s_\varepsilon(\tilde{y}, f)$ for $\tilde{x}, \tilde{y} \in M_f$ with $x_0 = y_0$.

For $\tilde{x} \in M_f$ the *stable* and the *unstable sets* are defined by

$$\begin{aligned} W^s(\tilde{x}, f) &= \{y \in M : \lim_{n\to\infty} d(x_n, f^n(y)) = 0\}, \\ W^u(\tilde{x}, f) &= \left\{ y \in M \;\middle|\; \begin{array}{l} \text{there is } \tilde{y} \in M_f \ \text{ satisfying } \ y_0 = y \\ \text{and } \lim_{n\to\infty} d(x_{-n}, y_{-n}) = 0 \end{array} \right\}. \end{aligned}$$

Then, $W^s(\tilde{x}, f) = W^s(\tilde{y}, f)$ for $\tilde{x}, \tilde{y} \in M_f$ with $x_0 = y_0$. If Λ is a hyperbolic set, for $\tilde{x} \in \Lambda_f$ we have

$$W^s(\tilde{x}, f) = \bigcup_{n=0}^{\infty} f^{-n}(W^s_\varepsilon(\tilde{f}^n(\tilde{x}), f)), \quad W^u(\tilde{x}, f) = \bigcup_{n=0}^{\infty} f^n(W^u_\varepsilon(\tilde{f}^{-n}(\tilde{x}), f)).$$

Remark that $W^\sigma(\tilde{x}, f)$ $(\sigma = s, u)$ are not always the manifolds like the stable and unstable manifolds given by diffeomorphisms. However we can define the transversality condition between $W^s(\tilde{x}, f)$ and $W^u(\tilde{y}, f)$ as follows.

Let $\tilde{y}$ and $\tilde{z}$ be points in Λ_f. We say that $W^s(\tilde{y}, f)$ is *transversal* to $W^u(\tilde{z}, f)$ if $f^{n+m} \mid W^u_\varepsilon(\tilde{f}^{-m}(\tilde{z}), f)$ is transversal to $W^s_\varepsilon(\tilde{f}^n(\tilde{y}), f)$ for $\varepsilon > 0$ small enough and $n, m \ge 0$.

The non-wandering set $\Omega(f)$ is defined by

$$\Omega(f) = \left\{ x \in M \;\middle|\; \begin{array}{l} \text{for any neighborhood } U \text{ of } x \text{ there is } n > 0 \\ \text{satisfying } U \cap f^n(U) \neq \emptyset \end{array} \right\}.$$

Obviously, $\Omega(f)$ is closed and satisfies that $f(\Omega(f)) \subset \Omega(f)$ and $Per(f) \subset \Omega(f)$, where $Per(f)$ denotes the set of all periodic points of f. A differentiable map f is said to satisfy *Axiom A* if

(i) $Per(f)$ is dense in $\Omega(f)$,

(ii) $\Omega(f)$ is hyperbolic.

We say that an Axiom A differentiable map f satisfies the *strong transversality* if $W^s(\tilde{y}, f)$ is transversal to $W^u(\tilde{z}, f)$ for $\tilde{y},\ \tilde{z} \in \Omega(f)_f$.

Theorem. If C^1-differentiable map f satisfies both Axiom A and the strong transversality, then $\tilde{f}$ satisfies C^1 uniformly shadowing property.

This result was proved by Sakai for the class of C^1-diffeomorphisms. The full proof of our theorem will appear elsewhere.

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